Linear impulse and momentum

Fact —

$$\textbf{Momentum} = \text{mass} \times \textbf{velocity}$$

Note that since *velocity* is a vector quantity, so is momentum.
In particular, in one dimension, this means that it is important to specify a direction.

Fact —

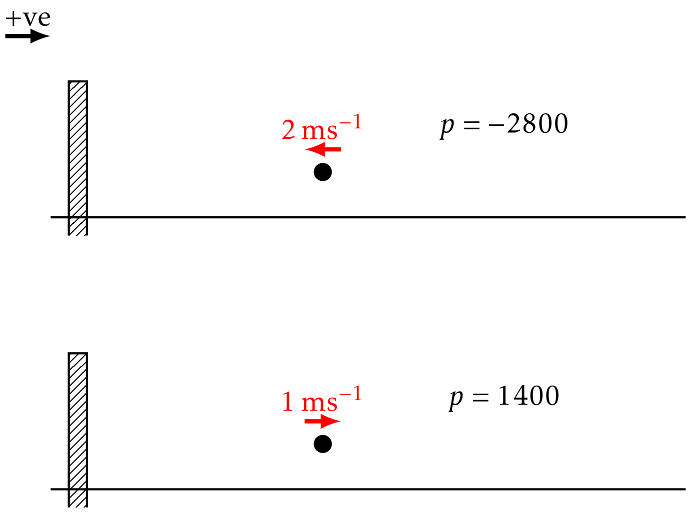
$$\textbf{Force} = \frac{d}{dt}(\textbf{momentum}) \quad \text{(N2)}$$

$$\textbf{Impulse} = \Delta\textbf{momentum}$$

$$\textbf{Impulse} = \textbf{Force} \times \text{time}$$

Example

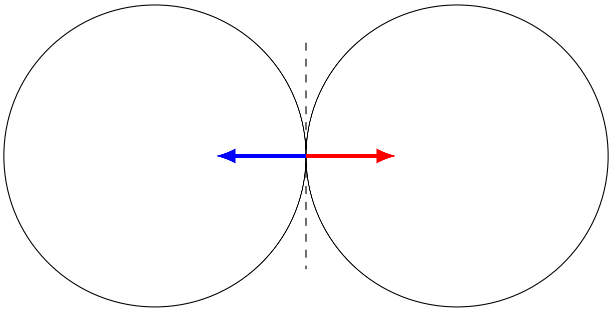
A truck of mass 1400 kg moving on a straight horizontal track at $2\,\text{ms}^{-1}$ runs into fixed buffers and rebounds at $1\,\text{ms}^{-1}$. The average force exerted by the buffers on the truck is 3500 N. How long were the buffers in contact with the truck?



$$\Delta p = 4200$$
$$Ft = \Delta p$$
$$\Rightarrow \quad t = \frac{4200}{3500}$$
$$= \frac{42}{35} = \frac{6}{5} = 1.25\,\text{s}$$

Collisions and the Principle of Conservation of Momentum

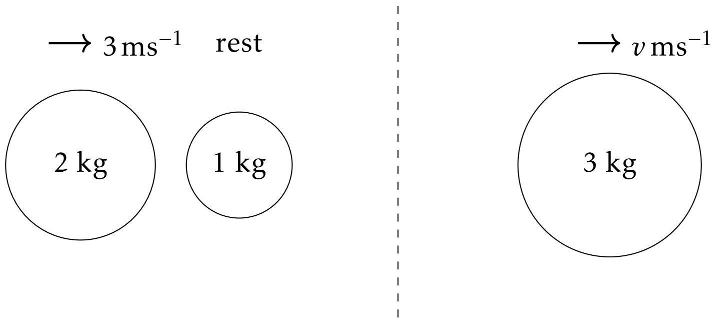
Fact (Newton's Third Law) — Every **action** has an equal and opposite **reaction**



By considering the impulse on the left particle, and the impulse on the right particle, we can see that the change in momentum of the left particle will be exactly offset by the change in momentum of the right particle.

Fact (Principle of Conservation of Momentum) —
For an **isolated system**, the total momentum of the system is constant.

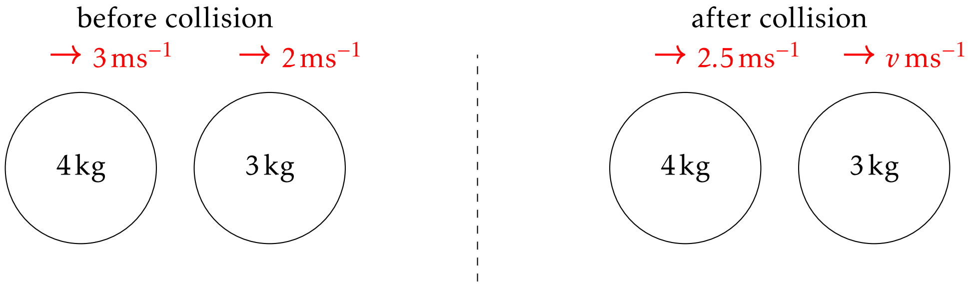
Example
A body of mass 2 kg moving on a smooth horizontal surface at $3\,\text{ms}^{-1}$, collides with a second body of mass 1 kg which is at rest. After the collision the bodies coalesce. Find the common speed of the bodies after impact.



$$\begin{aligned} \textit{initial momentum} &= 2\cdot 3 + 1\cdot 0 = 6 \\ \textit{final momentum} &= 3\cdot v = 3v \\ \textit{COM}: \quad 3v &= 6 \\ \Rightarrow \quad v &= 2\,\text{ms}^{-1} \end{aligned}$$

Example

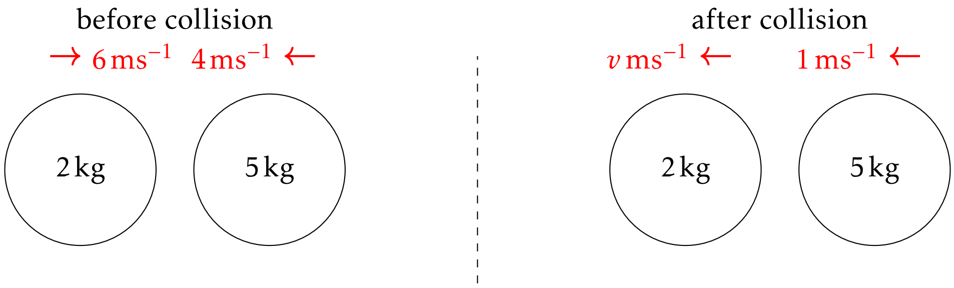
The two bodies shown collide on a smooth horizontal surface. Find the value v, the speed of the lighter body after impact.



$$
\begin{aligned}
COM: \quad & 4\cdot 3 + 2\cdot 3 = 4\cdot 2.5 + 3\cdot v \\
\Rightarrow \quad & 18 = 10 + 3v \\
\Rightarrow \quad & v = \frac{8}{3}\text{ms}^{-1}
\end{aligned}
$$

Example

The two bodies shown collide on a smooth horizontal surface. Find the value v, the speed of the lighter body after impact.

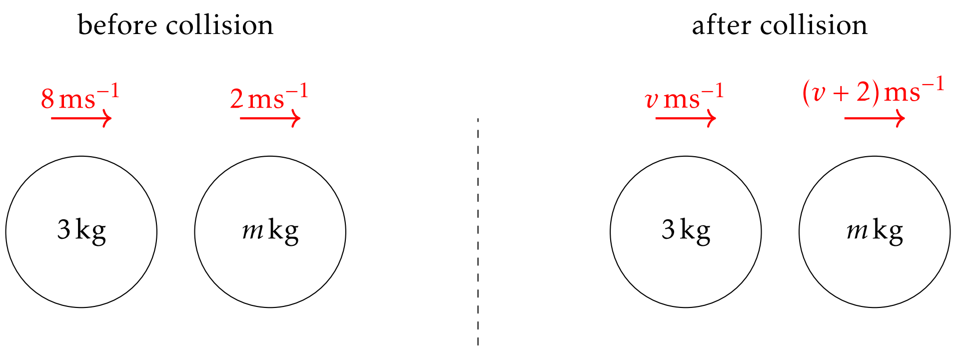


$$
\begin{aligned}
COM: \quad & 2\cdot 6 + 5\cdot(-4) = 2\cdot(-v) + 5\cdot(-1) \\
\Rightarrow \quad & -8 = -2v - 5 \\
\Rightarrow \quad & v = \frac{3}{2}\text{ms}^{-1}
\end{aligned}
$$

Example (Madas M1 Collisions Q9)

Two smooth spheres of equal radius, A and B , of mass 3 kg and $m$kg respectively, are moving in the same direction, along a straight line on a smooth horizontal plane. The spheres collide and the magnitude of impulse exerted on B by A is 15 Ns. Before the collision, the respective speeds of A and B are $8\,\text{ms}^{-1}$ and $2\,\text{ms}^{-1}$. After the collision B is moving with speed $2\,\text{ms}^{-1}$ relative to A.

Determine the value of m and the speed of B, after the collision.



$$
\begin{aligned}
COM: && 3\cdot 8 + m\cdot 2 &= 3v + m(u+2) \\
\Rightarrow && 24 + 2m &= 3v + mu + 2m \\
\Rightarrow && 24 &= 3v + mu && (1) \\
(I = \Delta p): && 15 &= m(v+2) - 2m \\
\Rightarrow && 15 &= mv && (2) \\
(2) \to (1): && 24 &= 3v + 15 \\
\Rightarrow: && v &= 3 \\
\Rightarrow && m &= 5
\end{aligned}
$$

Therefore $m = 5$ and the speed of B is $5\,\text{ms}^{-1}$.

Restitution

Fact (Newton's Experimental Law / Newton's Law of Restitution) —

$$\frac{\text{speed of separation of particles}}{\text{speed of approach of particles}} = e$$

e is called the *'coefficient of restitution'*

- $0 \leq e \leq 1$
- $e = 1$ - collision is perfectly elastic
- $e = 0$ - collision is perfectly inelastic

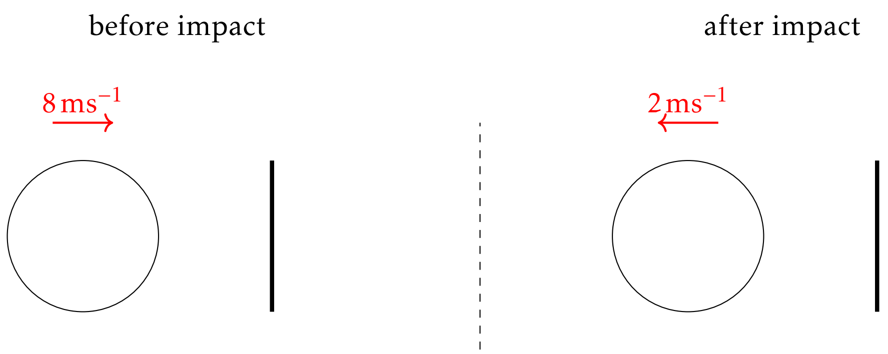
Tip (Separation is the opposite direction to approach!)

$$v_A - v_B = -e(u_A - u_B)$$

For particles with initial velocities $u_.$ and final velocities $v_.$ where the direction has been specified.

Example

A particle collides *normally* with a fixed vertical plane. The diagram shows the speeds of the particle before and after the collision. Find the value of the coefficient of restitution e



NEL :

$$e = \frac{\textit{speed of separation}}{\textit{speed of approach}}$$

$$= \frac{2}{8} = \frac{1}{4} \qquad = -\frac{-2-0}{8-0}$$

Therefore the coefficient of restitution is $\frac{1}{4}$.

What about ***conservation of momentum?***

Example

A particle falls 22.5 cm from rest onto a smooth horizontal plane. It then rebounds to a height of 10 cm. Find the coefficient of restitution between the particle and the plane. Give your answer to 2 s.f.

$$\begin{aligned} & & v^2 &= u^2 + 2as \\ &\Rightarrow & v^2 &= 0^2 + 2 \cdot g \cdot 0.225 \\ & & &= 4.41 \\ &\Rightarrow & v &= 2.1 \end{aligned}$$

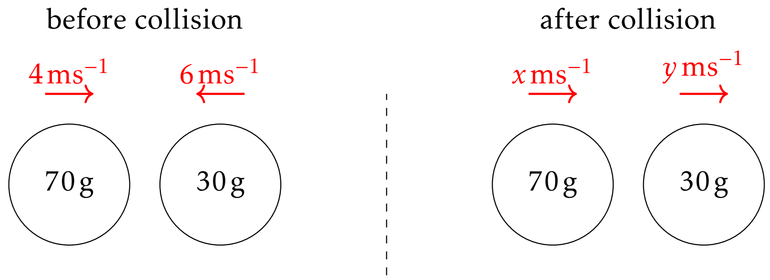
$$\begin{aligned} & & v^2 &= u^2 + 2as \\ &\Rightarrow & 0^2 &= u^2 - 2 \cdot g \cdot 0.1 \\ &\Rightarrow & u &= 1.4 \end{aligned}$$

$$NEL: \qquad e = \frac{1.4}{2.1} = \frac{2}{3}$$

Therefore the coefficient of restitution is 0.67 (2 *s.f.*)

Example

A metal ball of mass 70 g is moving at 4 ms^{-1}. It collides with a wooden ball of mass 30 g, which is moving in the same line in the opposite direction at 6 ms^{-1}. The coefficient of restitution is 0.5. Find the speeds of the balls after the collision.



$$\begin{aligned} NEL: & & \frac{x-y}{4-(-6)} &= -0.5 \\ \Rightarrow & & y &= x+5 \end{aligned}$$

$$\begin{aligned} COM: & & 70 \cdot 4 + 30 \cdot (-6) &= 70 \cdot x + 30 \cdot (x+5) \\ \Rightarrow & & 100 &= 100x + 150 \\ \Rightarrow & & x &= -0.5, y = 4.5 \end{aligned}$$

So both balls change directions. The metal ball rebounds at 0.5 ms^{-1}, *the wooden ball at* 4.5 ms^{-1}.

Example (OCR M2 June 2007 Q7)

Two small spheres A and B, with masses 0.3 kg and $m$kg respectively, lie at rest on a smooth horizontal surface. A is projected directly towards B with speed 6 ms^{-1} and hits B. The direction of motion of A is reversed in the collision. The speeds of A and B after the collision are 1 ms^{-1} and 3 ms^{-1} respectively. The coefficient of restitution between A and B is e.

(i) Show that $m = 0.7$. [2]

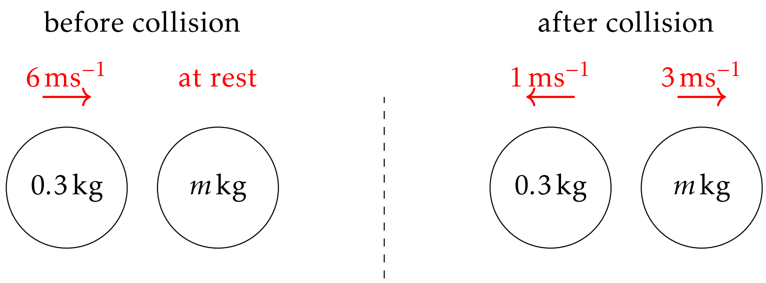
(ii) Find e. [2]

B continues to move at 3 ms^{-1} and strikes a vertical wall at right angles. The coefficient of restitution between B and the wall is f.

(iii) Find the range of values of f for which there will be a second collision between A and B. [2]

(iv) Find, in terms of f, the magnitude of the impulse that the wall exerts on B. [3]

(v) Given that $f = \frac{3}{4}$, calculate the final speeds of A and B, correct to 1 decimal place. [7]



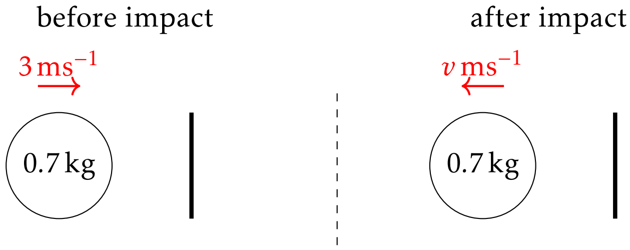
(i)

$$\text{COM}: \quad 0.3 \cdot 6 + m \cdot 0 = 0.3 \cdot (-1) + 3 \cdot m$$

$$\Rightarrow \quad m = \frac{1}{3}(0.3 \cdot 6 + 0.3) = 0.7$$

(ii)

$$\text{NEL}: \quad e = \frac{\text{speed of separation}}{\text{speed of approach}} = \frac{4}{6} = \frac{2}{3}.$$



(iii)

$$NEL: \quad f = \frac{\text{speed of separation}}{\text{speed of approach}} = \frac{v}{3}$$
$$\Rightarrow \quad v = 3f$$

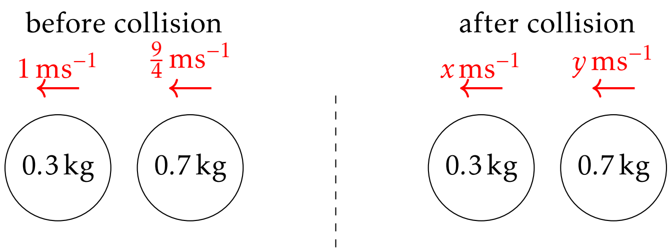
There will be another collision if $v > 1$ *(since the sphere on the left will be travelling faster than the sphere on the right. Therefore* $3f > 1 \Rightarrow f > \frac{1}{3}$*. Therefore* $\frac{1}{3} < f \leq 1$*. (Since coefficients of restitution are always between* 0 *and* 1*).*

(iv)

$$\begin{aligned} Impulse &= \Delta momentum \\ &= 0.7 \cdot (-v) - 0.7 \cdot 3 \\ &= 0.7 \cdot (-3f) - 0.7 \cdot 3 \\ &= -2.1(f+1) \end{aligned}$$

Therefore the magnitude is $2.1(1+f)$*. (And the direction is to the right (negative in our coordinate system).*

(v) *Since* $f = \frac{3}{4}$, $v = \frac{9}{4}$.



$$COM: \quad 0.3 \cdot 1 + 0.7 \cdot \frac{9}{4} = 0.3x + 0.7y$$
$$\Rightarrow \quad 3x + 7y = 18.75$$
$$NEL: \quad \frac{x-y}{1-\frac{9}{4}} = -\frac{2}{3}$$
$$\Rightarrow \quad x - y = \frac{5}{6}$$
$$\Rightarrow \quad x = \frac{59}{24}, y = \frac{13}{8}$$

Therefore A is travelling at $2.5\,\mathrm{ms}^{-1}$*, B is travelling at* $1.6\,\mathrm{ms}^{-1}$.

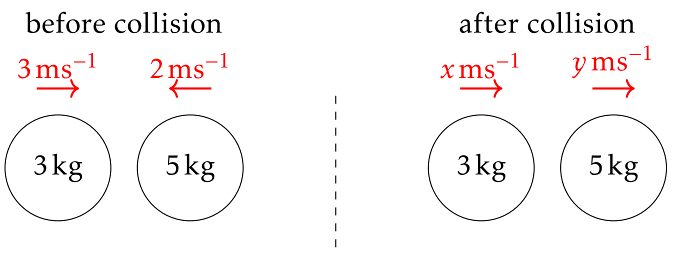
Energy in Collisions

Example

Two spheres A and B of equal radii have masses 3 kg and 5 kg respectively. A and B move towards each other along the same straight line on a smooth horizontal surface with velocities $3\,\text{ms}^{-1}$ and $2\,\text{ms}^{-1}$ respectively.

(a) If coefficient of restitution is $\frac{3}{5}$ find the velocities of the spheres after the collision.

(b) Find also the loss of kinetic energy due to the impact



(a)

$$\begin{aligned} COM: &\quad 3\cdot 3 + 5\cdot(-2) = 3x + 5y \\ \Rightarrow &\quad -1 = 3x + 5y \\ NEL: &\quad \frac{y-x}{5} = \frac{3}{5} \\ \Rightarrow &\quad 3 = y - x \\ \Rightarrow &\quad (x,y) = (-2,1) \end{aligned}$$

After the impact, the direction of A is reversed and its speed is $2\,\text{ms}^{-1}$. *The direction of B is also reversed, and its speed is* $1\,\text{ms}^{-1}$.

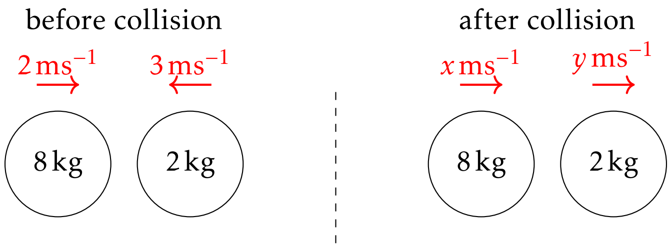
(b)

$$\begin{aligned} \text{Initial k.e.} &= \frac{1}{2}\cdot 3\cdot 3^2 + \frac{1}{2}\cdot 5\cdot(-2)^2 \\ &= 23.5\,\text{J} \\ \text{Final k.e.} &= \frac{1}{2}\cdot 3\cdot(-2)^2 + \frac{1}{2}\cdot 5\cdot 1^2 \\ &= 8.5\,\text{J} \\ \Rightarrow \quad \text{loss of k.e.} &= 15\,\text{J} \end{aligned}$$

What about conservation of energy?!

Example

A sphere of mass 8 kg moving at $2\,\mathrm{ms}^{-1}$ collides with another sphere of mass 2 kg moving at $3\,\mathrm{ms}^{-1}$ in the opposite direction. Find, in terms of the coefficient of restitution e, the loss of kinetic energy resulting from the collision.



The approach speed is 5*, so the separation speed is* $5e$*, ie* $y = x + 5e$.

$$
\begin{aligned}
\text{COM}: \quad 8\cdot 2 + 2\cdot(-3) &= 8\cdot x + 2\cdot(x+5e) \\
\Rightarrow \quad 10 &= 10x + 10e \\
\Rightarrow \quad x &= 1 - e \\
x + 5e &= 1 + 4e
\end{aligned}
$$

$$
\begin{aligned}
\text{initial k.e.} &= \frac{1}{2}\cdot 8\cdot 2^2 + \frac{1}{2}\cdot 2\cdot(-3)^2 \\
&= 25\,\mathrm{J} \\
\text{final k.e.} &= \frac{1}{2}\cdot 8\cdot(1-e)^2 + \frac{1}{2}\cdot 2\cdot(1+4e)^2 \\
&= 4(1-e)^2 + (1+4e)^2 \\
&= 4 - 8e + 4e^2 + 1 + 8e + 16e^2 \\
&= 5 + 20e^2 \\
\Rightarrow \quad \text{loss in k.e.} &= 25 - (5 + 20e^2) \\
&= 20(1-e^2)\,\mathrm{J}
\end{aligned}
$$